Received: *October 19, 1999*, Accepted: *March 12, 2000*

A path integral representation of the map between commutative and noncommutative gauge fields

Kazumi Okuyama

High Energy Accelerator Research Organization (KEK)
Tsukuba, Ibaraki 305-0801, Japan
E-mail: kazumi@post.kek.jp

Abstract: The world-volume theory on a D-brane in a constant B-field background can be described by either commutative or noncommutative Yang-Mills theories. These two descriptions correspond to two different gauge fixing of the diffeomorphism on the brane. Comparing the boundary states in the two gauges, we derive a map between commutative and noncommutative gauge fields in a path integral form, when the gauge group is U(1).

Keywords: D-branes, Gauge Symmetry, p-branes.

JHEP03(2000)016

Contents

1. Introduction

Noncommutativity of coordinates appears in the study of D-branes in two apparently different situations. One such situation occurs when N D-branes coincide. Then their transverse coordinates are promoted to $N \times N$ matrices. Another situation is the one in which the boundary coordinates $X^i(\tau)$ of an open string become noncommutative in the presence of a constant NS-NS B-field [1, 2, 3, 4]. The commutation relations of $X^i(\tau)$'s are written as

$$[X^i(\tau), X^j(\tau)] = i\theta^{ij} \,. \tag{1.1}$$

These relations lead to the noncommutativity of the world-volume coordinates of D-branes, which was first appeared in the compactification of Matrix theory in a three-form field background [5]–[9].

As was pointed out in [10], in a sense these two noncommutativity are "dual" to each other. A D-brane in a constant B-field background can be described as a collection of infinitely many lower dimensional D-branes [10, 11, 12, 13]. In this description, the transverse coordinates of lower dimensional D-branes should satisfy the same relation as (1.1).

In [14], Seiberg and Witten argued that the theory on D-branes in a B-field background can be described by either commutative or noncommutative Yang-Mills theories and these descriptions correspond to Pauli-Villars and point splitting regularizations of the world-sheet theory, respectively. They derived the relation between the commutative gauge field A and the noncommutative gauge field $\widehat{A}$ by requiring the equivalence of the gauge transformation of A and $\widehat{A}$.

In the D-brane world-volume perspective, these two descriptions correspond to two different gauge fixing of the world-volume diffeomorphism [15, 16]. One is the static gauge and the other is the "constant field strength gauge" (in the following, we will call the latter gauge "$\mathcal{F} = \omega$ gauge"). In the static gauge, the coordinates

JHEP03(2000)016

parallel to the brane are fixed and the ordinary gauge field remains as a dynamical field. In $\mathcal{F}=\omega$ gauge, the fluctuation of ordinary gauge field is set to zero. In this gauge, the dynamical degree of freedom are carried by the scalar fields corresponding to the parallel coordinates of the brane. The noncommutative gauge field appears as the fluctuation of this scalar field around the static gauge configuration. The two different descriptions are mapped to each other by the world-volume diffeomorphism.

In [15, 16], the relation between A and $\widehat{A}$ is derived from the diffeomorphism invariance of D-branes, but only in a semiclassical sense, i.e. the Moyal bracket is replaced with the Poisson bracket. To "quantize" the Poisson bracket into Moyal bracket, it is useful to use the path integral formalism [17, 18]. In this paper, we will derive the map $A \mapsto \widehat{A}$ in a path integral form by comparing the boundary states in the two different gauges, when the gauge group is U(1). The main claim of this paper is that the map is given by

$$\begin{aligned}\int \mathcal{D}\xi(\sigma)\, \exp\left(i\int d\sigma\, \frac{1}{2}\omega_{ij}\xi^i\partial_\sigma\xi^j + A_i(y+\xi)\partial_\sigma(y^i+\xi^i)\right) = \\ = \int \mathcal{D}\xi(\sigma)\, \exp\left(i\int d\sigma\, \frac{1}{2}\omega_{ij}\xi^i\partial_\sigma\xi^j + \widehat{A}_i(y+\xi)\partial_\sigma y^i\right),\end{aligned} \tag{1.2}$$

where $\omega_{ij}=(\theta^{-1})_{ij}$ and $y^i(\sigma)$ is an arbitrary function. We will show that this relation satisfies the requirement for the mapping between A and $\widehat{A}$, namely the equivalence of the gauge transformations for A and $\widehat{A}$.

Here we comment on the relation between (1.2) and the interpretation of the origin of noncommutativity in [14]. The left hand side of (1.2) is divergent due to the contraction of $\xi(\sigma)$ and $\partial_\sigma\xi(\sigma)$ at the same σ. We regularized it preserving the ordinary gauge invariance of A. On the other hand, the right hand side is finite because of the absence of this contraction. Therefore (1.2) naturally realizes the idea of [14], namely the commutative and noncommutative descriptions of gauge theory on D-branes correspond to the two different regularizations (point-splitting and Pauli-Villars) of the worldsheet theory.

This paper is organized as follows: In section 2, we review the symmetries of boundary states and the derivation of the map in the semiclassical form. In section 3, we derive the relation (1.2) by comparing the boundary states in the two different gauges and show that it gives the correct relation between A and $\widehat{A}$. We also argue how to regularize the left hand side of (1.2) without breaking the ordinary gauge invariance of A. Section 4 is devoted to discussions.

2. Symmetries of D-brane boundary states

In this section, we consider boundary states in a constant background B-field and their symmetries. In the following, we will gauge away the B-field in the bulk of the world-sheet and treat it as a gauge field background with constant field strength.

JHEP03(2000)016

To construct boundary states, it is convenient to introduce the coherent state $|x\rangle$ defined by $X^i(\sigma)|x\rangle = x^i(\sigma)|x\rangle$. $|x\rangle$ can be written as

$$|x\rangle = \exp\left(-i\int d\sigma P_i(\sigma)x^i(\sigma)\right)|D\rangle\,, \tag{2.1}$$

where $|D\rangle$ is the Dirichlet boundary state defined by $X^i(\sigma)|D\rangle = 0$ and P_i is the momentum conjugate to X^i. When we consider a Dp-brane, i runs from 0 to p. Using $|x\rangle$, the boundary state coupled to a U(1) gauge field $\mathcal{A}$ is given by [19]

$$|B\rangle = \int \mathcal{D}x e^{i\int \mathcal{A}(x)}|x\rangle\,. \tag{2.2}$$

In the case of the constant field strength $\mathcal{F}_{ij} = \omega_{ij}$, (2.2) is reduced to

$$|B\rangle = \int \mathcal{D}x \exp\left(i\int d\sigma \frac{1}{2}\omega_{ij}x^i\partial_\sigma x^j - P_i x^i\right)|D\rangle\,. \tag{2.3}$$

In the following, we assume that p is odd and ω_{ij} is invertible.

Now we consider the fluctuation of the Dp-brane around the above configuration. The most general form of the boundary state is

$$|\mathcal{A},\phi\rangle = \int \mathcal{D}\phi(x)\exp\left(i\int d\sigma(\mathcal{A}_i(x)\partial_\sigma x^i - P_i\phi^i(x))\right)|D\rangle\,, \tag{2.4}$$

where $\mathcal{A}_i$ is the gauge field on the Dp-brane and ϕ^i is the scalar field corresponding to the coordinate parallel to the Dp-brane. We suppress the transverse coordinates of the brane for simplicity. The fluctuation around the configuration (2.3) is parameterized as

$$\mathcal{A}_i = \frac{1}{2}\omega_{ji}x^j + A_i\,, \qquad \phi^i = x^i + \theta^{ij}a_j\,. \tag{2.5}$$

Following [15, 16], we review the argument that A_i and a_i become the ordinary and noncommutative gauge fields, respectively, after the gauge fixing of the diffeomorphism on the D-brane. So as to make $|\mathcal{A},\phi\rangle$ diffeomorphism invariant, we choose the measure $\mathcal{D}\phi(x)$ in (2.4) as

$$\mathcal{D}\phi(x) = \prod_\sigma dx(\sigma)\det\left(\frac{\partial\phi^i}{\partial x^j}\right)(x(\sigma))\,. \tag{2.6}$$

Under the diffeomorphism on the Dp-brane, $\mathcal{A}$ and ϕ^i transform as a 1-form and a scalar, respectively, i.e.

$$\begin{aligned}\delta_{\rm diff}\mathcal{A} &= \mathcal{L}_v\mathcal{A} = (di_v + i_v d)\mathcal{A}\,,\\ \delta_{\rm diff}\phi^i &= \mathcal{L}_v\phi^i = v^k\partial_k\phi^i\,.\end{aligned} \tag{2.7}$$

$|\mathcal{A},\phi\rangle$ is also invariant under the gauge transformation

$$\delta_{\rm gauge}\mathcal{A} = d\epsilon\,, \qquad \delta_{\rm gauge}\phi = 0\,, \tag{2.8}$$

JHEP03(2000)016

and the canonical transformation

$$\delta_{\rm can}\mathcal{A} = 0\,, \qquad \delta_{\rm can}\phi = \mathcal{L}_{{\rm ham}\lambda}\phi\,, \tag{2.9}$$

where hamλ is a hamiltonian vector field defined by

$$i_{{\rm ham}\lambda}\mathcal{F} = d\lambda\,. \tag{2.10}$$

We can see that the canonical transformation is equivalent to the field dependent gauge transformation up to the diffeomorphism:

$$\delta_{\rm can}(\lambda) = -\delta_{\rm gauge}(\lambda + i_{{\rm ham}\lambda}\mathcal{A}) + \delta_{\rm diff}({\rm ham}\lambda)\,. \tag{2.11}$$

By fixing the diffeomorphism invariance, we can obtain two different pictures for the same state. The first is the "static gauge" which is defined by $\phi^i = x^i$. In this gauge, $|\mathcal{A},\phi\rangle$ is reduced to

$$|\mathcal{A}\rangle = \int \mathcal{D}x e^{i\int \mathcal{A} - P_i x^i}|D\rangle\,. \tag{2.12}$$

The second is the "$\mathcal{F} = \omega$ gauge". In this gauge, the fluctuation of $\mathcal{A}$ is set to be zero, and $|\mathcal{A},\phi\rangle$ becomes

$$|\phi\rangle = \int \mathcal{D}\phi(x) e^{i\int \frac{1}{2}\omega_{ij}x^i dx^j - P_i\phi^i}|D\rangle\,. \tag{2.13}$$

For $|\mathcal{A}\rangle$, the residual diffeomorphism invariance is the canonical transformation with respect to the symplectic form $\mathcal{F}$ which coincides with the usual gauge symmetry for $\mathcal{A}$, or for A. On the other hand, $|\phi\rangle$ has no gauge field but it has a residual diffeomorphism symmetry which preserves the symplectic form ω. Its action on ϕ is given by

$$\delta\phi = \{\phi,\lambda\} = \theta^{kl}\partial_k\phi\partial_l\lambda\,. \tag{2.14}$$

In terms of a_i, this symmetry is written as $\theta^{ij}\delta a_j = \{\phi^i,\lambda\}$, or

$$\delta a_i = \partial_i\lambda + \theta^{kl}\partial_k a_i \partial_l\lambda\,. \tag{2.15}$$

Since the two states $|\mathcal{A}\rangle$ and $|\phi\rangle$ correspond to two different gauge choice for the same state, they should be equivalent under the diffeomorphism. Under the change of variable

$$x = \phi(y) \tag{2.16}$$

in the path integral (2.12), the equivalence of the two states (2.12) and (2.13) requires

$$\phi^*\mathcal{A} = \frac{1}{2}\omega_{ij}y^i dy^j + d(*)\,. \tag{2.17}$$

This relation gives a nontrivial mapping between A_i and a_i [15, 16].

Although one can show that a_i is equal to the noncommutative gauge field $\widehat{A}$ up to the second order in θ-expansion, a_i is obviously not equal to $\widehat{A}$ since the gauge transformation for a_i (2.15) is given in terms of the Poisson bracket instead of the Moyal bracket.

JHEP03(2000)016

3. The map between commutative and noncommutative gauge fields

In this section, we propose the resolution of the discrepancy between a_i and $\widehat{A}_i$ and give a simple rule for the map between A_i and $\widehat{A}_i$. As we will see below, in order to realize the noncommutative gauge symmetry for $\widehat{A}$, we should change the integration measure in $|\phi\rangle$ from $\mathcal{D}\phi(x)$ to the flat measure $\mathcal{D}x$. We write the state with the measure $\mathcal{D}x$ as

$$|\phi\rangle_{\rm NC} = \int \mathcal{D}x \exp\left(i\int d\sigma \frac{1}{2}\omega_{ij}x^i\partial_\sigma x^j - P_i\phi^i(x)\right)|D\rangle\,. \tag{3.1}$$

With this measure, the fluctuation of ϕ^i can be identified with $\widehat{A}$:

$$\phi^i(x) = x^i + \theta^{ij}\widehat{A}_j(x)\,. \tag{3.2}$$

To show that $|\phi\rangle_{\rm NC}$ is invariant under the noncommutative gauge transformation for $\widehat{A}$, it is convenient to use the T-dual picture. In terms of the coherent state $\widetilde{|y\rangle}$ for the T-dual coordinate, $|D\rangle$ is written as

$$|D\rangle = \int \mathcal{D}y\widetilde{|y\rangle}\,. \tag{3.3}$$

On $\widetilde{|y\rangle}$, P_i acts as $\partial_\sigma y_i$. Therefore, the original and the T-dual coherent state are related [20] by

$$|x\rangle = \int \mathcal{D}y \exp\left(-i\int d\sigma \partial_\sigma y_i x^i\right)\widetilde{|y\rangle}\,. \tag{3.4}$$

Using these relations, $|\phi\rangle_{\rm NC}$ can be written as

$$\begin{aligned}
|\phi\rangle_{\rm NC} &= \int \mathcal{D}x\mathcal{D}y \exp\left(i\int d\sigma \frac{1}{2}\omega_{ij}x^i\partial_\sigma x^j - \partial_\sigma y_i(x^i+\theta^{ij}\widehat{A}_j(x))\right)\widetilde{|y\rangle} \\
&= \int \mathcal{D}x\mathcal{D}y \exp\left(i\int d\sigma \frac{1}{2}\omega_{ij}(x^i-\theta^{ik}y_k)\partial_\sigma(x^j-\theta^{jl}y_l)+\right. \\
&\qquad\qquad \left. +\frac{1}{2}\theta^{ij}y_i\partial_\sigma y_j - \partial_\sigma y_i\theta^{ij}\widehat{A}_j(x)\right)\widetilde{|y\rangle} \\
&= \int \mathcal{D}\xi\mathcal{D}y \exp\left(i\int d\sigma \frac{1}{2}\omega_{ij}\xi^i\partial_\sigma\xi^j - \frac{1}{2}\omega_{ij}y^i\partial_\sigma y^j + \widehat{A}_i(\xi+y)\partial_\sigma y^i\right)\widetilde{|y\rangle}\,,
\end{aligned} \tag{3.5}$$

where $\xi^i = x^i - \theta^{ij}y_j$ and $y^i = \theta^{ij}y_j$. For notational simplicity, we introduce the quantity

$$\widehat{W}(\widehat{A}) = \left\langle \exp\left(i\int d\sigma \widehat{A}_i(y+\xi)\partial_\sigma y^i\right)\right\rangle_\xi\,, \tag{3.6}$$

where the expectation value is defined by

$$\langle\cdots\rangle_\xi = \int \mathcal{D}\xi\,(\cdots)e^{i\int d\sigma\frac{1}{2}\omega_{ij}\xi^i\partial_\sigma\xi^j}\,. \tag{3.7}$$

JHEP03(2000)016

Using $\widehat{W}(\widehat{A})$, eq. (3.5) is written as

$$|\phi\rangle_{\rm NC} = \int \mathcal{D}y\, \widehat{W}(\widehat{A}) \exp\left(-i \int d\sigma \frac{1}{2}\omega_{ij} y^i \partial_\sigma y^j\right) \widetilde{|y\rangle}\,. \tag{3.8}$$

In the same way, $|\mathcal{A}\rangle$ becomes

$$|\mathcal{A}\rangle = \int \mathcal{D}y\, W(A) \exp\left(-i \int d\sigma \frac{1}{2}\omega_{ij} y^i \partial_\sigma y^j\right) \widetilde{|y\rangle}\,, \tag{3.9}$$

with

$$W(A) = \left\langle \exp\left(i \int d\sigma A_i(y+\xi)\partial_\sigma(y^i+\xi^i)\right)\right\rangle_\xi\,. \tag{3.10}$$

The equivalence of the two descriptions $|\phi\rangle_{\rm NC}$ and $|\mathcal{A}\rangle$ requires

$$W(A) = \widehat{W}(\widehat{A})\,. \tag{3.11}$$

In the rest of this section, we will show that eq. (3.11) gives the correct mapping between the ordinary gauge field A and the noncommutative gauge field $\widehat{A}$.

What we have to show are:

1. $W(A)$ and $\widehat{W}(\widehat{A})$ have the same transformation property under the ordinary and noncommutative gauge transformations, respectively.
2. Eq. (3.11) has a nontrivial solution $\widehat{A}(A)$.

If these two conditions are satisfied, the map obtained from (3.11) agrees with the one defined in [14].

First let us consider the condition (1). The noncommutative gauge transformation for $\widehat{A}$ is

$$\delta \widehat{A}_i = \partial_i \widehat{\lambda} + i\widehat{\lambda} \star \widehat{A}_i - i\widehat{A}_i \star \widehat{\lambda}\,, \tag{3.12}$$

where the star product is defined by

$$f \star g(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{k_1 l_1} \cdots \theta^{k_n l_n} \partial_{k_1} \cdots \partial_{k_n} f(x) \partial_{l_1} \cdots \partial_{l_n} g(x)\,. \tag{3.13}$$

By expanding the exponential in $\widehat{W}(\widehat{A})$ and performing the Wick contraction of ξ using the two-point function

$$\langle \xi^i(\sigma)\xi^j(\sigma')\rangle_\xi = [(-i\omega\partial_\sigma)^{-1}]^{ij} = \frac{i}{2}\theta^{ij}\epsilon(\sigma-\sigma')\,, \tag{3.14}$$

$\widehat{W}(\widehat{A})$ can be rewritten as

$$\begin{aligned}\widehat{W}(\widehat{A}) &= 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d\sigma_1 \cdots d\sigma_n \mathbf{T}(\widehat{A}_{i_1} \star \cdots \star \widehat{A}_{i_n}) \partial_{\sigma_1} y^{i_1} \cdots \partial_{\sigma_n} y^{i_n} \\ &\equiv \mathbf{T} \exp_\star \left[i \int d\sigma \widehat{A}_i(y) \partial_\sigma y^i\right]\,,\end{aligned} \tag{3.15}$$

JHEP03(2000)016

where $\mathbf{T}$ denotes the time ordering. For example,

$$\mathbf{T}\Big\{f(y(\sigma_1))\star g(y(\sigma_2))\Big\} = \theta(\sigma_1-\sigma_2) f\star g + \theta(\sigma_2-\sigma_1) g\star f\,. \tag{3.16}$$

Note that (3.15) is the simplest example of the path integral representation of the star product studied in [17, 18]. In (3.15), we generalized the notion of the star product to the product of functions at different points:

$$f(x)\star g(y) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^{n}\theta^{k_1 l_1}\cdots\theta^{k_n l_n}\partial_{k_1}\cdots\partial_{k_n} f(x)\partial_{l_1}\cdots\partial_{l_n} g(y)\,. \tag{3.17}$$

From the expression (3.15), we can easily see that under the noncommutative gauge transformation (3.12) $\widehat{W}(\widehat{A})$ transforms as

$$\widehat{W}(\widehat{A}+\delta\widehat{A}) = e^{i\widehat{\lambda}(y(\sigma_f))}\star\widehat{W}(\widehat{A})\star e^{-i\widehat{\lambda}(y(\sigma_i))} \tag{3.18}$$

where we set the range of σ-integration to $[\sigma_i,\sigma_f]$.

Next we consider the transformation law of $W(A)$ under $\delta A_i = \partial_i\lambda$. Naively, $W(A)$ transforms as

$$\begin{aligned} W(A+d\lambda) &= \Bigg\langle \exp\Bigg(i\int_{\sigma_i}^{\sigma_f} d\sigma A_i(y+\xi)\partial_\sigma(y^i+\xi^i) + \\ &\qquad\qquad + i\lambda(y(\sigma_f)+\xi(\sigma_f)) - i\lambda(y(\sigma_i)+\xi(\sigma_i))\Bigg)\Bigg\rangle_{\xi} \\ &= e^{i\widehat{\lambda}(\lambda,A)(\sigma_f)}\star W(A)\star e^{-i\widehat{\lambda}(\lambda,A)(\sigma_i)}\,, \end{aligned} \tag{3.19}$$

where $\widehat{\lambda}(\lambda,A)$ is some function of λ and A whose explicit form is not needed in the following discussion. (3.19) shows that $W(A)$ has the same transformation property as that of $\widehat{W}(\widehat{A})$. But we should take care of the divergence coming from the contraction of ξ^i and $\partial_\sigma\xi^i$ with the same argument, since $\langle\xi^i(\sigma_1)\partial_{\sigma_2}\xi^j(\sigma_2)\rangle$ is proportional to the delta function $\delta(\sigma_1-\sigma_2)$. To make $W(A)$ finite, we regularize it by modifying the propagator of ξ:

$$W(A) = \int\mathcal{D}\xi\,\exp\left(i\int d\sigma\,\frac{1}{2}\omega_{ij}\xi^i M(\partial_\sigma,\Lambda)\xi^j + A_i(y+\xi)\partial_\sigma(y^i+\xi^i)\right), \tag{3.20}$$

with

$$M(\partial_\sigma,\Lambda) = \partial_\sigma - \frac{\partial_\sigma^3}{\Lambda^2}\,. \tag{3.21}$$

By the power counting, we can see that ∂_σ^3 term in M is sufficient to smear the delta-function singularity in $\langle\xi^i(\sigma_1)\partial_{\sigma_2}\xi^j(\sigma_2)\rangle$. Explicitly, the propagator is given by

$$\begin{aligned} \langle\xi^i(\sigma_1)\xi^j(\sigma_2)\rangle &= \frac{i}{2}\theta^{ij}\epsilon(\sigma_1-\sigma_2)\left(1-e^{-\Lambda|\sigma_1-\sigma_2|}\right) \\ \langle\xi^i(\sigma_1)\partial_{\sigma_2}\xi^j(\sigma_2)\rangle &= -i\theta^{ij}\frac{\Lambda}{2}e^{-\Lambda|\sigma_1-\sigma_2|}\,. \end{aligned} \tag{3.22}$$

JHEP03(2000)016

This regularization does not break the gauge covariance of $W(A)$ (3.19) since we do not change the second term of the exponent in (3.20).

Now we consider the condition (2). For the equation $W(A) = \widehat{W}(\widehat{A})$ to have a solution, $W(A)$ has to be the same form as $\widehat{W}(\widehat{A})$, namely the time ordered exponential. To explain that this is true, we expand $W(A)$ as

$$W(A) = \sum_{n=0}^{\infty} W_n(A)\,, \tag{3.23}$$

where $W_0(A) = 1$ and for $n > 1$ $W_n(A)$ is defined by

$$W_n(A) = \left\langle \frac{1}{n!} \left(i \int d\sigma A_i(y+\xi)\partial_\sigma (y^i + \xi^i) \right)^n \right\rangle_\xi . \tag{3.24}$$

We write the first few terms of the expansion of $W(A)$. $W_1(A)$ becomes

$$W_1(A) = \int d\sigma w_1(A) = \int d\sigma \Big[iA_i(y)\partial_\sigma y^i + w_1^{\mathrm{div}}(A,\Lambda) \Big]\,, \tag{3.25}$$

with

$$\lim_{\Lambda\to\infty} w_1^{\mathrm{div}}(A,\Lambda) = \frac{1}{2}\delta(0)\theta^{kl}F_{kl}(y)\,. \tag{3.26}$$

$W_2(A)$ becomes

$$W_2(A) = \int d\sigma w_2(A) + \frac{1}{2!}\int d\sigma_1 d\sigma_2 \mathbf{T}\Big[w_1(A(y(\sigma_1))) \star w_1(A(y(\sigma_2))) \Big]\,, \tag{3.27}$$

where

$$w_2(A) = iA_i^{(2)}\partial_\sigma y^i + w_2^{\mathrm{div}}(A,\Lambda) \tag{3.28}$$

and

$$\begin{aligned} A_i^{(2)} &= -\frac{1}{2}\theta^{kl}\{A_k \star (\partial_l A_i + F_{li})\}_S,\ \lim_{\Lambda\to\infty} w_2^{\mathrm{div}}(A,\Lambda) \\ &= -\frac{1}{4}\delta(0)\theta^{ij}\theta^{kl}F_{ik}F_{jl}\,. \end{aligned} \tag{3.29}$$

Here $\{\ \}_S$ in $A_i^{(2)}$ denotes the symmetrization of the star product

$$\{f_1 \star \cdots \star f_n\}_S = \frac{1}{n!}\sum_{\sigma\in S_n} f_{\sigma(1)} \star \cdots \star f_{\sigma(n)}. \tag{3.30}$$

We can deduce the structure of $W_n(A)$ without the knowledge of the explicit form of it. Taking account of the combinatorial factors of the Wick contraction, $W_n(A)$ becomes

$$W_n(A) = \sum_{\sum_m a_m k_m = n} \int \mathbf{T} \prod_m \frac{1}{a_m!} d\sigma_{i_1}\cdots d\sigma_{i_{a_m}} w_{k_m}(A(y(\sigma_{i_1})))\star\cdots\star w_{k_m}(A(y(\sigma_{i_{a_m}})))\,, \tag{3.31}$$

JHEP03(2000)016

where the summation is taken over the partition of n, and $w_n(A)$ has the form

$$w_n(A) = iA_i^{(n)}(y)\partial_\sigma y^i + w_n^{\rm div}(A,\Lambda). \tag{3.32}$$

$w_n^{\rm div}(A,\Lambda)$ is defined as the term in $w_n(A)$ which does not contain the factor $\partial_\sigma y$. Notice that $A_i^{(n)}$ is finite since the divergence in $w_n(A)$ comes from the contraction involving n $\partial_\sigma\xi$'s so the divergent term does not have the factor $\partial_\sigma y$.

Eq. (3.31) means that $W(A)$ has the form of the time ordered exponential, i.e.

$$W(A) = \sum_{n=0}^{\infty} W_n(A) = \mathbf{T}\exp_\star\left[\int d\sigma \sum_{n=1}^{\infty}\left(iA_i^{(n)}(y)\partial_\sigma y^i + w_n^{\rm div}(A,\Lambda)\right)\right]. \tag{3.33}$$

We can define the renormalized Wilson loop $W^{\rm fin}(A)$ by

$$W^{\rm fin}(A) = \mathbf{T}\exp_\star\left[\int d\sigma \sum_{n=1}^{\infty} iA_i^{(n)}(y)\partial_\sigma y^i\right]. \tag{3.34}$$

This renormalization can be achieved by adding the counter term to the path integral:

$$\begin{aligned} W^{\rm fin}(A) = \int \mathcal{D}\xi \exp\Bigg[\int d\sigma i\frac{1}{2}\omega_{ij}\xi^i M(\partial_\sigma,\Lambda)\xi^j + iA_i(y+\xi)\partial_\sigma(y^i+\xi^i) - \\ - \sum_{n=1}^{\infty}\left(w_n^{\rm div}(A(y+\xi),\Lambda) + w'_n(A(y+\xi),\Lambda)\right)\Bigg]. \end{aligned} \tag{3.35}$$

$w'_n(A,\Lambda)$ is the new counter term for the contraction between $\partial_\sigma\xi^i$ and $w_n^{\rm div}(A,\Lambda)$.

From (3.15) and (3.34), the solution of $W^{\rm fin}(A) = \widehat{W}(\widehat{A})$ is found to be

$$\widehat{A} = \sum_{n=1}^{\infty} A^{(n)}. \tag{3.36}$$

Here we list the first three terms of this expansion:

$$\begin{aligned} A_i^{(1)} &= A_i \\ A_i^{(2)} &= -\frac{1}{2}\theta^{kl}\{A_k\star(\partial_l A_i + F_{li})\}_S \\ A_i^{(3)} &= \frac{1}{2}\theta^{kl}\theta^{mn}\{A_k\star(\partial_l A_m\star\partial_n A_i - \partial_l F_{mi}\star A_n + F_{lm}\star F_{ni})\}_S. \end{aligned} \tag{3.37}$$

In principle, we can calculate the right hand side of (3.36) to any order in θ by the simple rule of the Wick contraction. It might be possible to write down the explicit form of the map to all order in θ in the same way as [17, 18], but we do not discuss it here.

We comment on the relation between our result and the argument of regularization in [14]. Since $\widehat{W}(\widehat{A})$ has the contraction between ξ's but does not contain the contraction of the form $\langle\xi(\sigma_1)\partial_\sigma\xi(\sigma_2)\rangle$, $\widehat{W}(\widehat{A})$ corresponds to the point-splitting

JHEP03(2000)016

regularization. On the other hand, $W(A)$ is regularized so as to preserve the covariance under the ordinary gauge symmetry. This is nothing but the mechanism of the appearance of the two descriptions of gauge theory on D-branes due to the possibility of two different regularizations. The only difference between our formalism and that in [14] is the dimension of the space on which the path integral is performed. In [14] the path integral is defined on the whole 2-dimensional worldsheet, while we consider that on the boundary of worldsheet.

4. Discussions

In this paper, we derived the map between A and $\widehat{A}$ in a path integral form by comparing the boundary states in two different gauges of the world-volume diffeomorphism. To realize the noncommutative gauge symmetry, we chose the flat measure $\mathcal{D}x$ in $|\phi\rangle_{\mathrm{NC}}$. This measure does not respect the canonical transformation symmetry of $|\phi\rangle$. This difference between $|\phi\rangle$ and $|\phi\rangle_{\mathrm{NC}}$ may be related to the "gauge equivalence of the star product" [17]. This viewpoint deserves further study.

We comment on the generalization of our result to $\mathrm{U}(N)$ gauge fields. One natural way to generalize (1.2) is to replace the exponential with the trace of path ordered exponential. But the relation obtained by this prescription is not enough to determine the map completely, since the gauge field has N^2 components. To construct the complete map, we may have to use the additional symmetry of the boundary state, such as the "non-abelian generalization of diffeomorphism" considered in [15].

Acknowledgments

I am grateful to N. Ishibashi and T. Kawano for discussions and encouragement, and K. Hayasaka, A. Alekseev and A. Bytsko for pointing out a mistake in the first version of this paper. I would also like to thank the organizers of the Summer Institute '99 in Fuji-Yoshida where this work started. This work was supported in part by JSPS Research Fellowships for Young Scientists.

References

[1] A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, *Open strings in background gauge fields*, *Nucl. Phys.* **B 280** (1987) 599.

[2] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219]; *Constrained quantization of open string in background b field and noncommutative D-brane*, hep-th/9906192.

[3] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Noncommutative geometry from strings and branes*, *J. High Energy Phys.* **02** (1999) 016 [hep-th/9810072]; *Dirac*

JHEP03(2000)016

quantization of open strings and noncommutativity in branes, hep-th/9906161;
M.M. Sheikh-Jabbari, *Open strings in a B-field background as electric dipoles*, *Phys. Lett.* **B 455** (1999) 129 [hep-th/9901080].

[4] V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[5] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[6] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[7] T. Kawano and K. Okuyama, *Matrix theory on noncommutative torus*, *Phys. Lett.* **B 433** (1998) 29 [hep-th/9803044].

[8] Y.-K.E. Cheung and M. Krogh, *Noncommutative geometry from 0-branes in a background B-field*, *Nucl. Phys.* **B 528** (1998) 185 [hep-th/9803031].

[9] P.-M. Ho, Y.-Y. Wu and Y.-S. Wu, *Towards a noncommutative geometric approach to matrix compactification*, *Phys. Rev.* **D 58** (1998) 026006 [hep-th/9712201];
P.-M. Ho and Y.-S. Wu, *Noncommutative gauge theories in matrix theory*, *Phys. Rev.* **D 58** (1998) 066003 [hep-th/9801147].

[10] N. Ishibashi, *p-branes from $(p-2)$-branes in the bosonic string theory*, *Nucl. Phys.* **B 539** (1999) 107 [hep-th/9804163].

[11] P.K. Townsend, *D-branes from m-branes*, *Phys. Lett.* **B 373** (1996) 68 [hep-th/9512062].

[12] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[13] T. Banks, N. Seiberg and S. Shenker, *Branes from matrices*, *Nucl. Phys.* **B 490** (1997) 91 [hep-th/9612157].

[14] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[15] N. Ishibashi, *A relation between commutative and noncommutative descriptions of D-branes*, hep-th/9909176.

[16] L. Cornalba and R. Schiappa, *Matrix theory star products from the Born-Infeld action*, hep-th/9907211;
L. Cornalba, *D-brane physics and noncommutative Yang-Mills theory*, hep-th/9909081.

[17] M. Kontsevich, *Deformation quantization of Poisson manifolds, 1*, q-alg/9709040.

JHEP03(2000)016

[18] A.S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, `math.QA/9902090`.

[19] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Loop corrections to superstring equations of motion*, *Nucl. Phys.* **B 308** (1988) 221.

[20] E. Alvarez, J.L. F. Barbon and J. Borlaf, *T-duality for open strings*, *Nucl. Phys.* **B 479** (1996) 218 [`hep-th/9603089`].

JHEP03(2000)016